Optimal frames for polarisation state reconstruction

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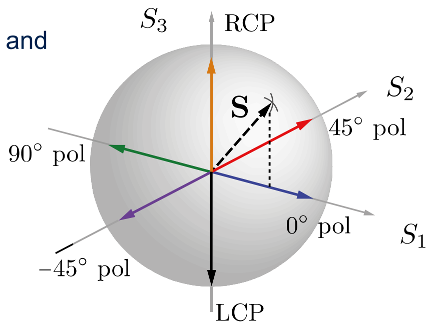
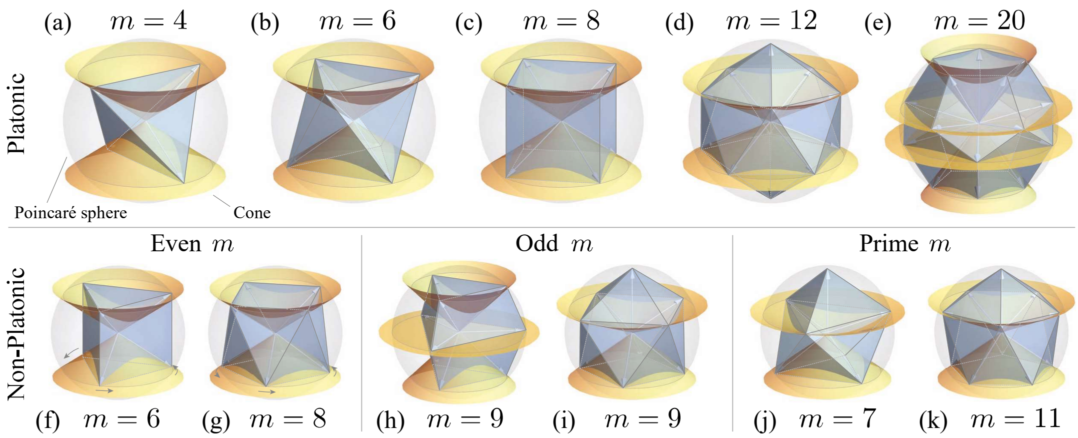
Introduction

- **Measurement of polarisation of light:** is a common problem in e.g. quantum information, astronomy, quantitative biology and single molecule orientational imaging.
- **Classical polarimetry:** Complete polarisation state reconstruction requires at least 4 projective measurements of the Stokes vector. Larger measurement frames can improve noise performance. We have derived optimal frames with an arbitrary number, *m*, of analysis states.
- **Nonlinear measurements and quantum state tomography:** polarisation dependent optical nonlinear processes can give insights into crystal and molecular structure, whilst higher order polarisation properties can contain “hidden” polarisation correlations, which are of interest both in a quantum and in a classical context. We present optimal measurement frames for reconstruction of such higher order properties.

Linear reconstruction and spherical designs

- **Linear measurement:** described by projection matrix $\mathbf{D} = \mathbb{A}\mathbf{S}$
- **Condition number:** characterises stability of inversion and bounds noise amplification $\kappa = \|\mathbb{A}\|\|\mathbb{A}^+\|$
- **Analytic minimisation:** equivalent to maximisation of determinant of Gram matrix
- **Optimal frame:** measurement states $\mathbf{A}_j = [1, \mathbf{a}_j]/2$ must satisfy: $\sum_{j=1}^m \mathbf{a}_j = \mathbf{0}$ and $\sum_{j=1}^m \mathbf{a}_j\mathbf{a}_j^T = \frac{m}{N-1}\mathbb{I}_{N-1}$
- **Spherical 2 designs:** fulfil all optimality constraints





Properties of optimal bases

- Optimal condition number is independent of *m*.
- Non-unique in general and non-existent for *m* = 5.
- For *m* = 4, 6, 8, 12 and 20 can be defined by the vertices of the Platonic solids inscribed in the Poincaré sphere.
- Do not necessarily correspond to inscribed polyhedra of maximum volume
- Also optimal in sense of equally weighted variance

Generalisation to N dimensional measurements

- *N* = 3: linear polarimetry
- *N* = 9: polarimetry of 3D optical fields
- $\kappa^2 = 2N^2 - 4N + 4$
- $\mathrm{EWV} = 2\kappa^2/m$

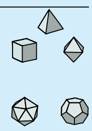
Higher order polarisation properties

- **Nonlinear measurands:** dependent on products of Stokes parameters upto degree *t*
  $$D(\mathbf{S}) = \sum_{k=0}^{K} F_k P_k(\mathbf{S})$$
- **Sum rule:** Polynomial functions satisfy a sum rule
  $$\sum_{k=0}^{K} P_k^*(\mathbf{S}) P_k(\mathbf{S}) = C$$
- **Minimum condition number:** follows from extension of the optimisation strategy. Worsens for higher order problems.
  $$\kappa^2 = \frac{C}{P_0} + \frac{CK^2}{C - P_0^2},$$
- **Spherical 2*t* designs:** Optimal measurement frames correspond to spherical 2*t* designs. These include the Platonic solids.

Spherical T designs

Integral of any polynomial of degree *t* over a sphere = the average over the set of *m* points

	exists iff
2-design	$m = 4, \geq 6$
3-design	$m = 6, 8, \geq 10$
4-design	$m = 12, 14, \geq 20$
5-design	$m = 12, 16, 17, 20, \geq 22$
6-design	$m = 24, 26, \geq 28$
7-design	$m = 24, 30, 32, 34, \geq 36$
8-design	$m = 36, 40, 42, \geq 44$
9-design	$m = 48, 50, 52, \geq 54$
10-design	$m = 60, 62, \geq 64$



- Generalisation of mutually unbiased bases

References

M. R. Foreman, A. Favaro and A. Aiello, Phys. Rev. Lett **115**, 263901 (2015).
R. H. Hardin and N. J. A. Sloane, Discrete Comput. Geom **15**, 429–441 (1996).
Y. Mimura, Graphs Combi. **6**, 369–372 (1990).